MATH 2230 HW2

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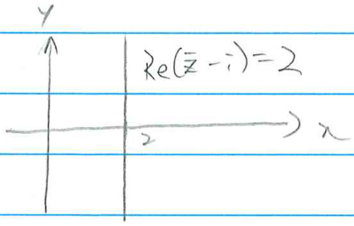
1(a) $\overline{\bar{z}+3i} = \bar{\bar{z}} + \overline{3i} = z - 3i$

(b) $\overline{iz} = \bar{i}\,\bar{z} = -i\bar{z}$

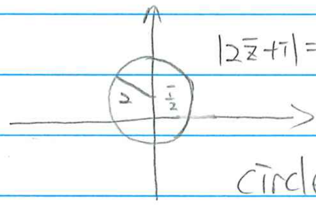
(c) $\overline{(2+i)^2} = (\overline{2+i})^2 = (2-i)^2 = (4-1) + (-2-2)i = 3-4i$

(d) $|(2\bar{z}+5)(\sqrt{2}-i)| = |\sqrt{2}-i|\,|\overline{2z+5}| = \sqrt{3}\,|2z+5|$

2(a)



(b)



$|2\bar{z}+i| = 4$

$|2\bar{z}+i| = 4$

$\Rightarrow |z - \frac{i}{2}| = 2$

circle centre at $\frac{i}{2}$ with radius 2

9. $$\left|\frac{1}{z^4-4z^2+3}\right| = \frac{1}{|z^2-3|\,|z^2-1|} \le \frac{1}{||z|^2-3|\,||z|^2-1|} = \frac{1}{(2^2-3)(2^2-1)} = \frac{1}{3}$$

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1(a) $2i = 2e^{\frac{\pi}{2}}$

$\therefore$ square roots of $2i$:

$\sqrt{2}\, e^{\frac{\frac{\pi}{2}+2n\pi}{2}}$, $n = 0, 1$

$= \sqrt{2}\, e^{\frac{m\pi}{4}}$, $m = 1$ or 5

$= \pm\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$= \pm(1+i)$

rect. coord.: $(1,1), (-1,-1)$

(b) $1-\sqrt{3}i = 2\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right) = 2e^{-\frac{\pi}{3}}$

$\therefore$ square roots of $1-\sqrt{3}i$:

$\sqrt{2}\, e^{\frac{-\frac{\pi}{3}+2n\pi}{2}}$, $n = 0, 1$

$= \sqrt{2}\, e^{\frac{m\pi}{6}}$, $m = -1, 5$

$= \pm\sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= \pm\frac{\sqrt{3}-i}{\sqrt{2}}$

rect. coord.: $\left(\frac{\sqrt{3}}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right), \left(-\frac{\sqrt{3}}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

2. $-8i = 8e^{\frac{3\pi}{2}}$

cubic roots: $2e^{\frac{\frac{3\pi}{2}+2n\pi}{3}}$, $n = 0, 1, 2$

$= 2e^{\frac{m\pi}{6}}$, $m = 3, 7, 11$

$= 2i$ or $2\left(\pm\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= 2i$ or $\pm\sqrt{3} - i$

$\therefore$ $c_1 = 2i$

$c_2 = -\sqrt{3} - i$

$c_3 = \sqrt{3} - i$

8(a)
$$az^2 + bz + c = 0$$
$$a(z^2 + \frac{b}{a}z) + c = 0$$
$$a(z^2 + \frac{b}{a}z + (\frac{b}{2a})^2 - (\frac{b}{2a})^2) + c = 0$$
$$a(z + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = 0$$
$$(z + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$
$$z + \frac{b}{2a} = \left(\frac{b^2 - 4ac}{4a^2}\right)^{\frac{1}{2}}$$
$$z = \frac{-b + (b^2 - 4ac)^{\frac{1}{2}}}{2a}$$

(b)
$$z = \frac{-2 + (2^2 - 4(1-i))^{\frac{1}{2}}}{2}$$
$$= \frac{-2 + (4i)^{\frac{1}{2}}}{2}$$
$$= \frac{1}{2}(-2 + (4e^{\frac{\pi}{2}i})^{\frac{1}{2}})$$
$$= -1 + e^{\frac{\frac{\pi}{2} + 2n\pi}{2}i}, \quad n = 0, 1$$
$$= -1 + e^{\frac{\pi}{4}i} \quad \text{or} \quad -1 + e^{\frac{5\pi}{4}i}$$
$$= -1 + (\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i) \quad \text{or} \quad -1 + (-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)$$
$$= (-1 + \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}}i \quad \text{or} \quad (-1 - \frac{1}{\sqrt{2}}) - \frac{1}{\sqrt{2}}i$$

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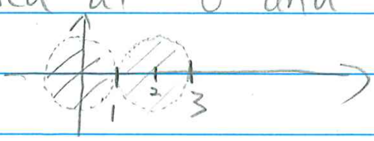
5. S consists of 2 disjoint open disk of radius 1, centred at 0 and 2 respectively. (Their boundary intersect only at the point 1, but S does not contain the boundary)



8. Let E be such set. Let z be a boundary point. Suppose $z \notin E$. By def. of z, every neighbourhood of z contain at least 1 point belonging to E, which is not equal to z. Then by def., z is an accumulation point of E, but E contains all accumulation point. Contradiction. So $z \in E$.

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2(a) $z^3+z+1 = (x+iy)^3+(x+iy)+1$

$= x^3+3x^2yi+3x(yi)^2+(yi)^3+x+iy+1$

$= (x^3-3xy^2+x+1)+i(3x^2y-y^3+y)$

(b) $f(z) = \frac{(\bar{z})^2}{z}$

$= \frac{(x-iy)^2}{x+iy}$

$= \frac{(x-iy)^3}{x^2+y^2}$

$= \frac{x^3-3x^2yi-3xy^2+iy^3}{x^2+y^2}$

$= \frac{x^3-3xy^2}{x^2+y^2}+i\frac{y^3-3x^2y}{x^2+y^2}$

4. $f(z) = z+\frac{1}{\bar{z}}$

$= re^{i\theta}+\frac{1}{r}e^{-i\theta}$, for $z=re^{i\theta}$

$= r(\cos\theta+i\sin\theta)+\frac{1}{r}(\cos\theta-i\sin\theta)$

$= (r+\frac{1}{r})\cos\theta+i(r-\frac{1}{r})\sin\theta$

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1(a) Let $\varepsilon>0$. Let $\delta=\varepsilon$.

$|Re(z)-Re(z_0)| \le |z-z_0| < \varepsilon$ if $0<|z-z_0|<\delta$. So it's true.

(b) Let $\varepsilon>0$. Let $\delta=\varepsilon$

$|\bar{z}-\bar{z}_0| = |z-z_0| < \varepsilon$ if $0<|z-z_0|<\delta$. So it's true.

(c) Let $\varepsilon>0$. Let $\delta=\varepsilon$.

$\left|\frac{\bar{z}^2}{z}-0\right| = \frac{|z|^2}{|z|} = |z| < \varepsilon$ if $0<|z-0|<\delta$. So it's true.

3(a) $\lim_{z\to z_0} \frac{1}{z^n} = \frac{1}{\lim_{z\to z_0} z^n} = \frac{1}{z_0^n}$

(b) $\lim_{z\to i} \frac{iz^3-1}{z+i} = \frac{\lim_{z\to i} iz^3-1}{\lim_{z\to i} z+i} = \frac{i^4-1}{2i} = 0$

(c) $\lim_{z\to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z\to z_0} P(z)}{\lim_{z\to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}$

5. On the real axis, $f(z) = \left(\frac{x}{x}\right)^2 = 1$
On the imaginary axis, $f(z) = \left(\frac{-y}{y}\right)^2 = 1$
On $y=x$, $f(z) = \left(\frac{x-ix}{x+ix}\right)^2 = \left(\frac{1-i}{1+i}\right)^2 = \left(\frac{-2i}{2}\right)^2 = -1$
The limit approaching 0 on the real axis and on the line $y=x$ does not equal, so the limit does not exists.

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1(a) $\exp(2\pm 3\pi i) = e^2 e^{\pm 3\pi i} = -e^2$

(b) $\exp\left(\frac{2+\pi i}{4}\right) = e^{\frac{1}{2}} e^{\frac{\pi}{4}i} = e^{\frac{1}{2}}\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \frac{\sqrt{e}}{\sqrt{2}}(1+i)$

(c) $\exp(z+\pi i) = \exp(z)e^{\pi i} = -\exp z$

10 let $z = x+iy$
(a) $e^z = e^{x+iy} = e^x(\cos y + i\sin y)$ is real
iff $\sin y = 0$ iff $y = n\pi$ for $n\in\mathbb{Z}$

(b) $e^z = e^x(\cos y + i\sin y)$ is pure imaginary
iff $\cos y = 0$ iff $y = (n+\frac{1}{2})\pi$ for $n\in\mathbb{Z}$

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1(a) $Log(-ei) = Log(e \cdot e^{-\frac{\pi}{2}i}) = \ln e - \frac{\pi}{2}i = 1 - \frac{\pi}{2}i$

(b) $Log(1-i) = Log(\sqrt{2}(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i)) = Log(\sqrt{2}e^{-\frac{\pi}{4}i}) = \ln\sqrt{2} - \frac{\pi}{4}i = \frac{1}{2}\ln 2 - \frac{\pi}{4}i$

2(a) $\log e = \log e e^{2n\pi i}$, $n \in \mathbb{Z}$

$= 1 + 2n\pi i$, $n \in \mathbb{Z}$

(b) $\log i = \log e^{(2n+\frac{1}{2})\pi i}$, $n \in \mathbb{Z}$

$= (2n+\frac{1}{2})\pi i$, $n \in \mathbb{Z}$

(c) $\log(-1+\sqrt{3}i) = \log(2(-\frac{1}{2} + \frac{\sqrt{3}}{2}i))$

$= \ln 2 + \log e^{(2n+\frac{2}{3})\pi i}$, $n \in \mathbb{Z}$

$= \ln 2 + 2(n+\frac{1}{3})\pi i$, $n \in \mathbb{Z}$

5(a) $i = e^{\frac{\pi}{2}i}$. So $(e^{\frac{\pi}{2}i})^{\frac{1}{2}} = e^{\frac{\pi}{4}i}$ or $e^{\frac{\frac{\pi}{2}+2\pi}{2}}$

$= e^{\frac{\pi}{4}i}$ or $e^{\frac{5\pi}{4}}$

$\log(e^{i\frac{\pi}{4}}) = \log e^{i(\frac{\pi}{4}+2n\pi)}$, $n \in \mathbb{Z}$

$= (2n+\frac{1}{4})\pi i$

$\log(e^{i\frac{5\pi}{4}}) = \log e^{i(\frac{5\pi}{4}+2n\pi)}$

$= (2n+\frac{5}{4})\pi i$

$= ((2n+1)+\frac{1}{4})\pi i$, $n \in \mathbb{Z}$

$\log(i^{\frac{1}{2}}) = \log(e^{i\frac{\pi}{4}})$ or $\log e^{i\frac{5\pi}{4}}$

$= (2n+\frac{1}{4})\pi i$ or $((2n+1)+\frac{1}{4})\pi i$, $n \in \mathbb{Z}$

$= (n+\frac{1}{4})\pi i$, $n \in \mathbb{Z}$

(b) $\log i = \log e^{(2n\pi+\frac{\pi}{2})i}$

$= (2n+\frac{1}{2})\pi i$

$= 2(n+\frac{1}{4})\pi i$, $n \in \mathbb{Z}$

$= 2\log i^{\frac{1}{2}}$